Motivation for risk-neutral pricing *ansatz* (single-branch example)

Limitation of "constancy of asset price ratios" criterion that can be used when final prices are deterministic

The simple condition

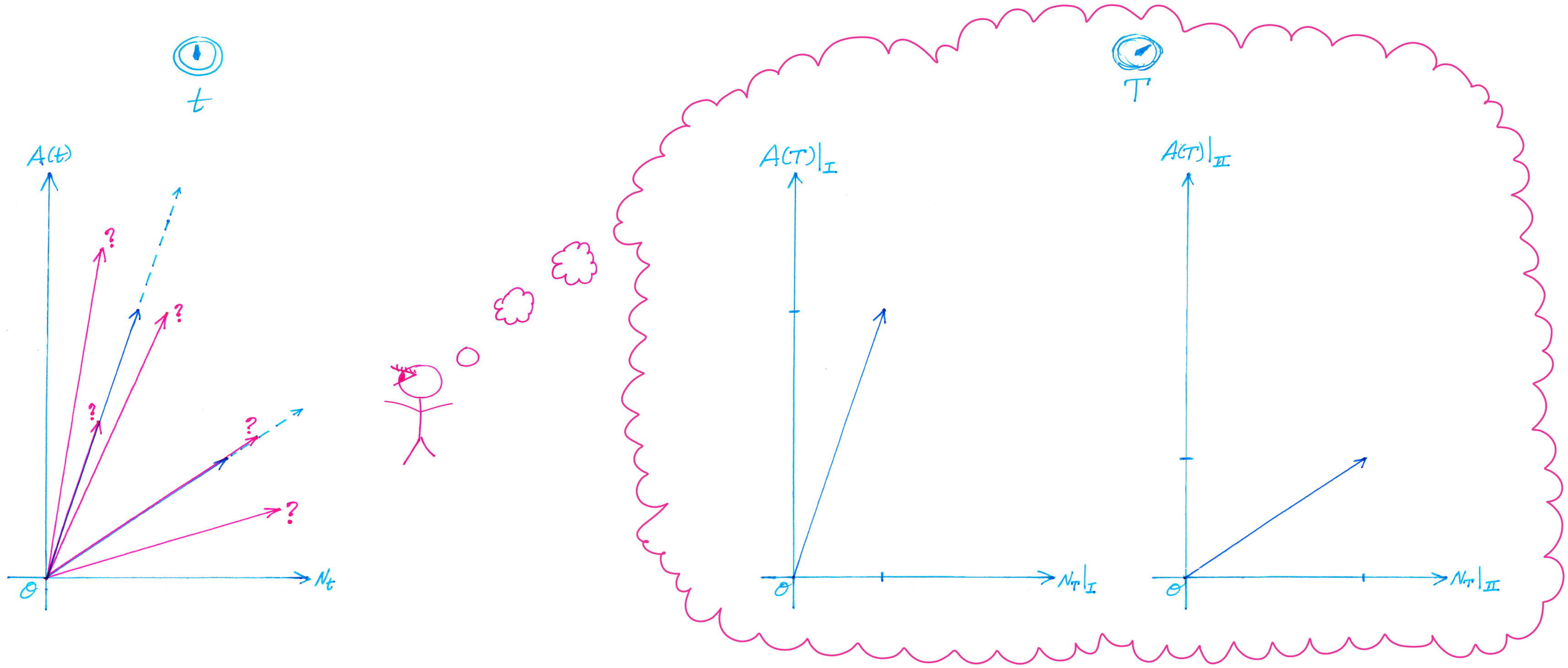
$$\frac{A(t)}{N_t} = \frac{A(T)}{N_T}$$

cannot be satisfied when

$$\frac{A(T)}{N_T}$$

has different values in different final states.

In the following toy example, the investigator at present time t thinks that there are exactly two final outcome states, I and II, that can be realized at later time T. The investigator thinks that each of these final states has a finite probability of being realized at time T.



Motivation for risk-neutral pricing *ansatz* (single-branch example)

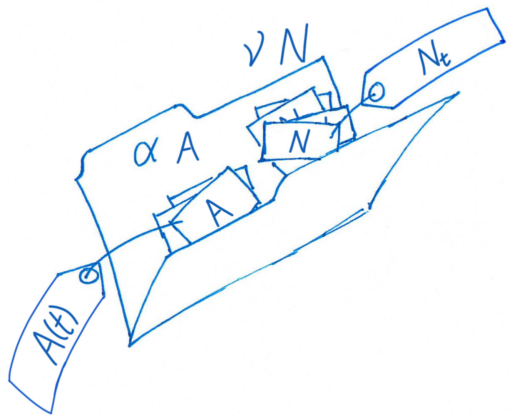
Arbitrage portfolio (a little more generalized)

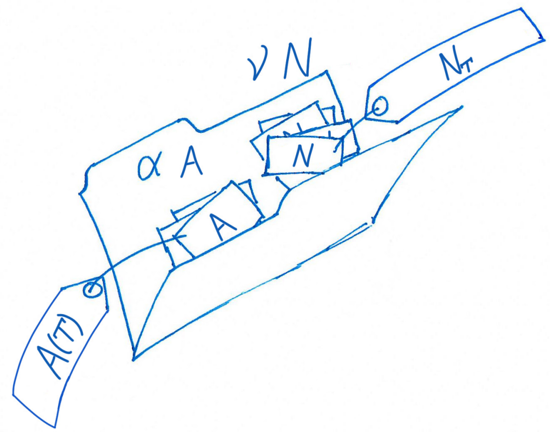
	Mathematical	English phrase
1.	$V(t) = 0$	For free
2.	$V(T) \geq 0$	Guarantee of no loss
3.	$P(V(T) > 0) > 0$	Finite probability of gain

A portfolio satisfying criteria 1, 2, and 3 is a more general example of an **arbitrage portfolio.**

Portfolio

Asset	Units	Unit price at time t	Assumptions
A	α	$A(t)$	$A(t), A(T) > 0$ (for this motivating example)
N	ν	N_t	$N_t, N_T > 0$ (keep this constraint in general)





Use criterion 1 to constrain composition of portfolio:

Initial portfolio price	Final portfolio price
$V(t) = \alpha A(t) + \nu N_t = 0$ $= \nu N_t \left[\frac{\alpha}{\nu}\frac{A(t)}{N_t} + 1\right] = 0$ $\frac{\alpha}{\nu} = -\frac{1}{\left(\frac{A(t)}{N_t}\right)}$	$V(T) = \alpha A(T) + \nu N_T$ $= \nu N_T \left[\frac{\alpha}{\nu}\frac{A(T)}{N_T} + 1\right]$ $= \nu N_T \left[-\frac{\left(\frac{A(T)}{N_T}\right)}{\left(\frac{A(t)}{N_t}\right)} + 1\right]$

Motivation for risk-neutral pricing *ansatz* (single-branch example)

Trying to construct arbitrage portfolios in different pricing scenarios

	Case	Final portfolio price in state I	Final portfolio price in state II	Strategy
(a)	$\left.\frac{A(T)}{N_T}\right\|_{II} < \left.\frac{A(T)}{N_T}\right\|_{I} < \frac{A(t)}{N_t}$	$V(T)\|_I = \nu N_T\|_I \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_I}{\left(\frac{A(t)}{N_t}\right)}}_{<1} + 1 \right]}_{>0}$	$V(T)\|_{II} = \nu N_T\|_{II} \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_{II}}{\left(\frac{A(t)}{N_t}\right)}}_{<1} + 1 \right]}_{>0}$	Set $\nu > 0$
(b)	$\left.\frac{A(T)}{N_T}\right\|_{II} < \left.\frac{A(T)}{N_T}\right\|_{I} = \frac{A(t)}{N_t}$	$V(T)\|_I = \nu N_T\|_I \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_I}{\left(\frac{A(t)}{N_t}\right)}}_{1} + 1 \right]}_{0}$	$V(T)\|_{II} = \nu N_T\|_{II} \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_{II}}{\left(\frac{A(t)}{N_t}\right)}}_{<1} + 1 \right]}_{>0}$	Set $\nu > 0$
(c)	$\left.\frac{A(T)}{N_T}\right\|_{II} < \frac{A(t)}{N_t} < \left.\frac{A(T)}{N_T}\right\|_{I}$	$V(T)\|_I = \nu N_T\|_I \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_I}{\left(\frac{A(t)}{N_t}\right)}}_{>1} + 1 \right]}_{<0}$	$V(T)\|_{II} = \nu N_T\|_{II} \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_{II}}{\left(\frac{A(t)}{N_t}\right)}}_{<1} + 1 \right]}_{>0}$	No strategy available
(d)	$\frac{A(t)}{N_t} = \left.\frac{A(T)}{N_T}\right\|_{II} < \left.\frac{A(T)}{N_T}\right\|_{I}$	$V(T)\|_I = \nu N_T\|_I \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_I}{\left(\frac{A(t)}{N_t}\right)}}_{>1} + 1 \right]}_{<0}$	$V(T)\|_{II} = \nu N_T\|_{II} \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_{II}}{\left(\frac{A(t)}{N_t}\right)}}_{1} + 1 \right]}_{0}$	Set $\nu < 0$
(e)	$\frac{A(t)}{N_t} < \left.\frac{A(T)}{N_T}\right\|_{II} < \left.\frac{A(T)}{N_T}\right\|_{I}$	$V(T)\|_I = \nu N_T\|_I \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_I}{\left(\frac{A(t)}{N_t}\right)}}_{>1} + 1 \right]}_{<0}$	$V(T)\|_{II} = \nu N_T\|_{II} \underbrace{\left[-\underbrace{\frac{\left.\frac{A(T)}{N_T}\right\|_{II}}{\left(\frac{A(t)}{N_t}\right)}}_{>1} + 1 \right]}_{<0}$	Set $\nu < 0$

Motivation for risk-neutral pricing *ansatz* (single-branch example)

Risk-neutral pricing *ansatz*

More general **no-arbitrage assumption:** Opportunities to construct arbitrage portfolios are quickly consumed by arbitrageurs.

Only pricing satisfying case (c) can be abundantly sustained.

$$\left.\frac{A(T)}{N_T}\right|_{II} < \frac{A(t)}{N_t} < \left.\frac{A(T)}{N_T}\right|_{I}$$

Betweenness can be expressed using a weighted average.

$$1 \cdot \frac{A(t)}{N_t} = p_A^* \cdot \left.\frac{A(T)}{N_T}\right|_{I} + (1 - p_A^*) \cdot \left.\frac{A(T)}{N_T}\right|_{II}$$

$$0 < p_A^* < 1$$

For every asset X with prices $X(t), X(T) > 0$, it must be that

$$\frac{X(t)}{N_t} = p_X^* \left.\frac{X(T)}{N_T}\right|_{I} + (1 - p_X^*) \left.\frac{X(T)}{N_T}\right|_{II}$$

$$0 < p_X^* < 1$$

Risk-neutral pricing *ansatz*

Suppose asset N ("numeraire") has prices $N_t, N_T > 0$. Guess that for all assets $X \in \{A, B, \dots\}$,

$$\frac{X(t)}{N_t} = p_X^* \left.\frac{X(T)}{N_T}\right|_{I} + (1 - p_X^*) \left.\frac{X(T)}{N_T}\right|_{II}$$

$$0 < p_X^* < 1$$

with the stipulation that

$$p_A^* = p_B^* = \cdots = p^*$$

The common weighting coefficient p^* is called the **risk-neutral "probability"**.